power spectral density functions are replaced with their aliased periodic counterparts. Evaluating these periodic functions, which are usually expressed as infinite sums of periodically shifted spectral densities, is simplified by a method based on contour integration. The example using a second-order innovations representation demonstrated the simplicity of this technique. As with the nonaliased case, the aliased Wiener filter requires complete knowledge of the second-order statistics of the processes of interest. In practice, we would model these processes using such a (factorable) representation from which we could obtain the necessary second-order statistics. Since the noncausal solution discussed herein is unrealizable, we would need to implement a causal form of the filter either by truncating and delaying the noncausal form or by solving the causal estimation problem through the use of factorization techniques such as that proposed by Bode and Shannon [4].

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Noise Amplification of Periodic Nonuniform Sampling

Daniel Seidner and Meir Feder

***Abstract*—In this correspondence, we discuss periodic nonuniform sampling as described by Yen. Such a sampling scheme is more sensitive to additive white noise than uniform sampling. We give here an explicit formula for the noise amplification of periodic nonuniform sampling compared with uniform sampling.**

***Index Terms*—Nonuniform sampling, quantization, signal reconstruction, signal sampling.**

I. Introduction

It is well known that uniform sampling is less sensitive to additive white noise than other forms of sampling such as derivative sampling and "bunched" or periodic nonuniform sampling (see Bracewell [1], for example). A good review of sensitivity issues can be found in Marks [2]. In this correspondence, we give a quantitative measure to describe the noise sensitivity of periodic nonuniform sampling compared with uniform sampling.

We discuss the case of a bandlimited signal $f(t)$ with bandwidth $2B$, i.e., its Fourier transform $F(\omega)$ is zero for $|\omega| > B$. We sample this signal so that the average rate equals the Nyquist rate of B/π samples/s. In every period of $M\pi/B$ seconds, we have M samples at times $\tau_1, \tau_2, \cdots, \tau_M$ relative to the beginning of the period. This scheme was first described by Yen [3], who also provided the explicit interpolation formula.

The analysis conducted by Yen assumed an exact knowledge of the value of the samples. In practice, we never have the exact samples value due to quantization and noise. Thus, an interesting question is the quality of the reconstruction in the presence of noise. We specifically discuss the case where the measurement noise is independent of the signal and white.

When adding a white quantization noise to the samples, we have a reconstruction error $v^r(t)$. Its variance, which is denoted $E\{|v^r(t)|^2\}$, depends on the sampling scheme.

It is convenient to represent the periodic nonuniform sampling according to Papoulis' generalized sampling expansion (GSE) [4]. In a GSE system, the input is fed to M filters. To assure reconstruction, the M outputs of these filters can be sampled simultaneously at $1/M$th of the Nyquist rate, i.e., the sampling period is $M\pi/B$. Choosing the M filters to be time shift elements with

$$H_k(\omega) = e^{j\omega\tau_k} \tag{1}$$

where $H_k(\omega)$ is the Fourier transform of the kth filter and where $k = 1, 2, \cdots, M$ results in a GSE system that is equivalent to the nonuniform sampling scheme described above.

II. Analysis

We start our analysis by adding a zero mean discrete stochastic noise sequence to each of the sample sequences resulting from sampling the

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M output channels of the GSE system, prior to reconstruction. We denote the nth noise value, which is added to the nth sample of the kth channel, as $v_k(nT)$, where belowdisplayskip20pt

$$E\{v_k(nT)v_q^*(mT)\} = \sigma_v^2 \delta_{k,\,q}\delta_{n,\,m} \tag{2}$$

and $v_k(nT)$ is a zero mean random variable. This is equivalent to adding a zero mean discrete stochastic white noise sequence with variance σ_v^2 to the sequence resulting from the periodic nonuniform sampling. We define the noise amplification factor A_ϵ as belowdisplayskip20pt

$$A_\epsilon = \frac{\overline{E\{|v^r(t)|^2\}}}{\sigma_v^2} \tag{3}$$

where $\overline{E\{|v^r(t)|^2\}}$ is the time average of $E\{|v^r(t)|^2\}$, and A_ϵ is equal to Marks' normalized interpolation noise variance (NINV) [2].

We use the analysis of white additive noise in GSE systems as developed in [5]. In this analysis, we found that belowdisplayskip20pt

$$\begin{aligned} A_\epsilon &= \frac{1}{c}\int_{-B}^{-B+c} \sum_{k=1}^{M}\sum_{l=1}^{M} |T_{kl}^{-1}(\omega)|^2\, d\omega \\ &= \frac{1}{c}\int_{-B}^{-B+c} \text{tr}\{\mathbf{T}(\omega)^{-1^T}\mathbf{T}(\omega)^{-1^*}\}\, d\omega \end{aligned} \tag{4}$$

where $\mathbf{T}(\omega)$ is Papoulis' GSE matrix, which is an $M \times M$ matrix whose $(k,\, l)$th component is given by belowdisplayskip20pt

$$T_{kl}(\omega) = H_k(\omega + (l-1)\cdot c) \tag{5}$$

and where $c = 2B/M$. A similar expression has been derived by Cheung and Marks in [6].

In our case, $H_k(\omega) = e^{j\omega\tau_k}$; thereforebelowdisplayskip20pt

$$T_{kl}(\omega) = e^{j\omega\tau_k}e^{j(l-1)c\tau_k} \tag{6}$$

or

$$\mathbf{T}(\omega) = \mathbf{B}(\omega)\mathbf{A} \tag{7}$$

where belowdisplayskip20pt

$$\mathbf{B}(\omega) = \text{diag}\{e^{j\omega\tau_1},\, e^{j\omega\tau_2},\, \cdots,\, e^{j\omega\tau_M}\} \tag{8}$$

and

$$\mathbf{A} = \begin{bmatrix} 1 & e^{jc\tau_1} & e^{j2c\tau_1} & \cdots & e^{j(M-1)\tau_1} \\ 1 & e^{jc\tau_2} & e^{j2c\tau_2} & \cdots & e^{j(M-1)\tau_2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & e^{jc\tau_M} & e^{j2c\tau_M} & \cdots & e^{j(M-1)\tau_M} \end{bmatrix}. \tag{9}$$

Since $\text{tr}\{\mathbf{T}(\omega)^{-1^T}\mathbf{T}(\omega)^{-1^*}\} = \text{tr}\{\mathbf{A}^{-1^T}\mathbf{A}^{-1^*}\}$, we only need to find $\mathbf{A}^{-1}$ in order to calculate the noise amplification factor. Note that $\mathbf{A}$ is a Vandermonde matrix for which the formula for the inverse is known (see e.g., [7] or [8]). Here, we use a slightly different representation. Let $x_k = e^{jc\tau_k}$. The $(k,\, l)$th component of $\mathbf{A}$ is belowdisplayskip15pt

$$A_{kl} = x_k^{(l-1)} \tag{10}$$

and the matrix $\mathbf{A}$ therefore becomes

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{(M-1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{(M-1)} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & x_M & x_M^2 & \cdots & x_M^{(M-1)} \end{bmatrix} \tag{11}$$

which is the classic Vandermonde matrix. Therefore

$$A_{kl}^{-1} = \frac{a_{l,k-1}}{\prod_{\substack{i=1\\ i\neq l}}^{M}(x_l - x_i)} \tag{12}$$

where $\{a_{l,\,k-1}\}_{k=1}^{M}$ are the coefficients of the polynomial $P_l(x)$

$$P_l(x) = \prod_{\substack{i=1\\ i\neq l}}^{M}(x - x_i) = \sum_{q=0}^{M-1} a_{l,\,q}x^q. \tag{13}$$

Now

$$\begin{aligned} \text{tr}\{\mathbf{A}^{-1^T}\mathbf{A}^{-1^*}\} &= \sum_{l=1}^{M}\sum_{k=1}^{M} |A_{kl}^{-1}|^2 \\ &= \sum_{l=1}^{M}\sum_{k=1}^{M} \left| \frac{a_{l,k-1}}{\prod_{\substack{i=1\\ i\neq l}}^{M}(x_l - x_i)} \right|^2 \\ &= \sum_{l=1}^{M} \frac{\sum_{k=1}^{M} |a_{l,k-1}|^2}{\prod_{\substack{i=1\\ i\neq l}}^{M}|(x_l - x_i)|^2}. \end{aligned} \tag{14}$$

We find the sum $\sum_{k=1}^{M}|a_{l,\,k-1}|^2$ using Parseval theorem. Consider the discrete Fourier transform (DFT) of the sequence $\{a_{l,\,0},\, a_{l,\,1},\, \cdots,\, a_{l,\,M-1}\}$. The kth component $\tilde{A}_{l,\,k}$ of this DFT is

$$\tilde{A}_{l,\,k} = \sum_{q=0}^{M-1} a_{l,\,q}e^{-j(2\pi/M)kq} = P_l(e^{-j(2\pi/M)k}) \qquad k = 0,\, \cdots,\, M-1. \tag{15}$$

According to the Parseval theorem, we have

$$\begin{aligned} \sum_{k=1}^{M} |a_{l,\,k-1}|^2 &= \frac{1}{M}\sum_{k=0}^{M-1} |\tilde{A}_{l,\,k}|^2 \\ &= \frac{1}{M}\sum_{k=0}^{M-1} |P_l(e^{-j(2\pi/M)k})|^2 \\ &= \frac{1}{M}\sum_{k=0}^{M-1} |P_l(e^{j(2\pi/M)k})|^2 \\ &= \frac{1}{M}\sum_{k=0}^{M-1}\prod_{\substack{i=1\\ i\neq l}}^{M} |(e^{j(2\pi/M)k} - x_i)|^2. \end{aligned} \tag{16}$$

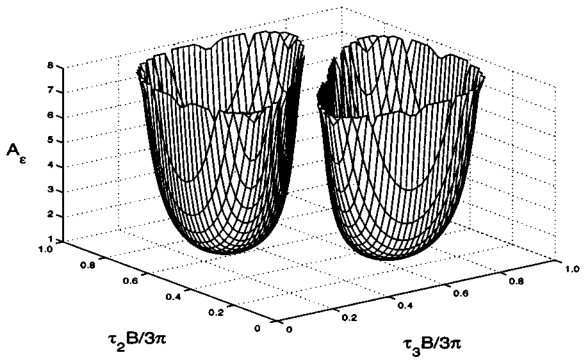

Fig. 1. Noise amplification of periodic nonuniform sampling $(M = 3)$.

Therefore, we can write

$$\begin{aligned} A_\epsilon &= \frac{1}{c}\int_{-B}^{-B+c} \mathrm{tr}\{\mathbf{T}(\omega)^{-1^T}\mathbf{T}(\omega)^{-1^*}\}\,d\omega \\ &= \frac{1}{c}\int_{-B}^{-B+c} \mathrm{tr}\{\mathbf{A}^{-1^T}\mathbf{A}^{-1^*}\}\,d\omega \\ &= \frac{1}{c}\int_{-B}^{-B+c} \sum_{l=1}^{M}\sum_{k=1}^{M} |A_{kl}^{-1}|^2\,d\omega \\ &= \frac{1}{c}\int_{-B}^{-B+c} \sum_{l=1}^{M} \frac{\sum_{k=1}^{M}|a_{l,k-1}|^2}{\prod_{\substack{i=1\\ i\neq l}}^{M}|(x_l - x_i)|^2}\,d\omega \\ &= \frac{1}{c}\int_{-B}^{-B+c} \sum_{l=1}^{M} \frac{\frac{1}{M}\sum_{k=0}^{M-1}\prod_{\substack{i=1\\ i\neq l}}^{M}|(e^{j(2\pi/M)k} - x_i)|^2}{\prod_{\substack{i=1\\ i\neq l}}^{M}|(x_l - x_i)|^2}\,d\omega. \end{aligned} \tag{17}$$

Using the relations $x_i = e^{jc\tau_i}$, $|e^{jc\tau_1} - e^{jc\tau_2}|^2 = 4\sin^2(c/2)(\tau_1 - \tau_2)$, and $e^{j(2\pi/M)k} = e^{jc(\pi/B)k}$, we have

$$A_\epsilon = \frac{1}{M}\sum_{l=1}^{M} \frac{\sum_{k=0}^{M-1}\prod_{\substack{i=1\\ i\neq l}}^{M}\sin^2\frac{c}{2}\left(k\frac{\pi}{B} - \tau_i\right)}{\prod_{\substack{i=1\\ i\neq l}}^{M}\sin^2\frac{c}{2}(\tau_l - \tau_i)}. \tag{18}$$

This equation gives the noise amplification as a function of the τ_i's that define the nonuniform sampling. We see that if we choose $\tau_i = (i-1)\pi/B$, which means uniform sampling, we get the minimum noise amplification $A_\epsilon = 1$, i.e., $\overline{E\{|v^r(t)|^2\}} = \sigma_v^2$. As shown in [5], uniform sampling is an optimal sampling scheme.

Fig. 1 describes A_ϵ of a periodic nonuniform sampling with $M = 3$ and $\tau_1 = 0$ as a function of τ_2 and τ_3, which change from 0 to $3\pi/B$.

III. Conclusion

We have found the explicit formula of the noise amplification for periodic nonuniform sampling. This formula gives a quantitative measure to the amount of reconstruction noise in periodic nonuniform sampling created by white quantization noise, compared with uniform sampling, which is an optimal sampling scheme. The result in this correspondence is an interesting particular case of the sensitivity analysis of [5].

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Wideband Beamspace Adaptive Array Utilizing FIR Fan Filters for Multibeam Forming

Takashi Sekiguchi and Yoshio Karasawa

***Abstract*—We propose a wideband beamspace adaptive array that uses FIR fan filters to construct a multibeam forming network. We also describe a method for designing such FIR fan filters. Approximation is achieved by combining spectral transformation and the window method such that the beam patterns including the sidelobe characteristics of the resulting fan filters are virtually frequency independent. This is a requirement of beamforming networks used in multibeam forming. Fan filters designed with the proposed method are used to demonstrate that the beam-space adaptive array can suppress interference signals having a wide fractional bandwidth and that the array has fast convergence.**

***Index Terms*—Adaptive arrays, array processing, fan filters, multidimensional digital filters.**

I. Introduction

In the fields of radio communications, radar, and acoustics, wideband adaptive beamforming is an important technique for rejecting interference signals whose incident direction into a sensor array differs from that of the desired signals.

Tapped-delay line circuits with adaptive coefficients are often used for wideband adaptive arrays [1]. In general, however, they require

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